

Electromagnetic Angular Momentum and Relativity

Kimball A. Milton*

H. L. Dodge Department of Physics and Astronomy, University of Oklahoma, Norman, OK 73019

Giulio Meille†

Department of Physics, Columbia University, New York, NY 10027

(Dated: August 24, 2012)

Recently there have been suggestions that the Lorentz force law is inconsistent with special relativity. This is difficult to understand, since Einstein invented relativity in order to reconcile electrodynamics with mechanics. Here we investigate the momentum of an electric charge and a magnetic dipole in the frame in which both are at rest, and in an infinitesimally boosted frame in which both have a common velocity. We show that for a dipole composed of a magnetic monopole-antimonopole pair the torque is zero in both frames, while if the dipole is a point dipole, the torque is not zero, but is balanced by the rate of change of the angular momentum of the electromagnetic field, so there is no mechanical torque on the dipole.

PACS numbers: 03.50.De, 03.30.+p, 41.20.-q, 14.80.Hv

INTRODUCTION

Considerable press was given to a recent claim by Mansuripur [1] that the Lorentz force law appears to be in contradiction with special relativity. Because that force law is based on Maxwell's equations, which are the origin and basis for Einsteinian relativity, it is difficult to understand this claim. Mansuripur's analysis, based on Ref. [2], suggests that the Lorentz force law must be replaced by some other expression, such as that of Einstein and Laub [3, 4]. This conclusion has received widespread publicity in the popular press [5]

By now there have appeared many critiques of this conclusion [6–12]. The consensus seems firm that omitted contributions remove the contradiction. However, these analyses seem to consider special cases and somewhat obscure concepts, such as hidden momentum [13, 14], so it seems to us that a simple general analysis will clarify all points and settle the issue definitively.

The problem posed by Mansuripur is the following. Consider an electric charge and a magnetic dipole separated by some distance at rest. Evidently, there is no force or torque on either particle. When the same system is viewed in a boosted frame, where both particles have a common velocity, it appears that the torque on the dipole

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B} + \mathbf{d} \times \mathbf{E} \quad (1)$$

is nonzero. Thus in this frame it seems there should be precession of the dipole, violating the principle of relativity.

In this note we revisit this problem in the following way. We first compute the momentum and angular momentum of the electromagnetic field for the static configuration of charge and dipole; the values of these quantities are different for a point dipole and a dipole composed of a magnetic monopole-antimonopole pair. This bears

on the issue of “hidden momentum.” Then we calculate the charge and current densities, and the fields, in the boosted frame, and directly compute the torque on the dipole from the microscopic Lorentz force law. This torque identically vanishes for a dipole composed of a magnetic monopole-antimonopole, but is nonzero for a point dipole. However, the latter is precisely cancelled by the nonzero rate of change of the angular momentum of the electromagnetic field in this case, so in either event, there is no mechanical torque on the dipole.

MOMENTUM AND ANGULAR MOMENTUM OF CHARGE AND DIPOLE

In this section we consider a static system consisting of a charge e , which we may take at the origin, and a dipole $\boldsymbol{\mu}$ at position \mathbf{R} , in vacuum. We will calculate the field momentum and angular momentum for this configuration from the formulas (throughout this paper we use Gaussian units; our primary reference is Ref. [15])

$$\mathbf{P} = \int (d\mathbf{r}) \mathbf{G}, \quad \mathbf{G} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}, \quad (2a)$$

$$\mathbf{J} = \int (d\mathbf{r}) \mathbf{r} \times \mathbf{G}. \quad (2b)$$

Let us first adopt the model (“Gilbert”) in which the magnetic dipole consists of a magnetic monopole of charge $+g$ separated by a displacement \mathbf{a} from a monopole of charge $-g$. We will always suppose that $a \ll R$. The momentum of a system consisting of one electric charge e and one magnetic pole g is

$$\mathbf{P}_{eg} = \frac{eg}{4\pi c} \int (d\mathbf{r}) \frac{\mathbf{r}}{r^3} \times \frac{\mathbf{r} - \mathbf{R}}{|\mathbf{r} - \mathbf{R}|^3}. \quad (3)$$

Now the following dyadic (and convergent) integral is eas-

ily worked out:

$$\int (d\mathbf{r}) \frac{\mathbf{r}}{r^3} \frac{\mathbf{r} - \mathbf{R}}{|\mathbf{r} - \mathbf{R}|^3} = \frac{2\pi}{R} \left(\mathbf{1} - \frac{\mathbf{R}\mathbf{R}}{R^2} \right), \quad (4)$$

from which we immediately conclude that the momentum $\mathbf{P}_{eg} = 0$. From this it follows that the momentum of the charge and a Gilbert dipole is also zero,

$$\mathbf{P}_{eg\bar{g}} = 0. \quad (5)$$

The same integral (4) allows us to conclude that the angular momentum of a charge and a magnetic monopole is nonzero,

$$\mathbf{J}_{eg} = -\frac{eg}{4\pi c} \int (d\mathbf{r}) \mathbf{r} \times \frac{\mathbf{r} \times \mathbf{R}}{r^3 |\mathbf{r} - \mathbf{R}|^3} = \frac{eg}{c} \frac{\mathbf{R}}{R}, \quad (6)$$

a result first discovered by Poincaré [16] and then by Thomson [17]. From this the angular momentum for a dipole made from a monopole-antimonopole separated by a distance \mathbf{a} is

$$\mathbf{J}_{eg\bar{g}} = \frac{eg}{c} \mathbf{a} \cdot \nabla \frac{\mathbf{R}}{R} = \frac{1}{c} \mathbf{R} \times (\boldsymbol{\mu} \times \mathbf{E}(\mathbf{R})), \quad (7)$$

where $\boldsymbol{\mu} = g\mathbf{a}$ and $\mathbf{E} = e\mathbf{R}/R^3$.

Now consider a point magnetic dipole, which possesses the magnetic field

$$\mathbf{B} = \mathbf{B}_s + \mathbf{B}_f, \quad (8)$$

where the second term is the usual field of a magnetic dipole,

$$\begin{aligned} \mathbf{B}_f &= -\nabla \left(\boldsymbol{\mu} \cdot \frac{\mathbf{r} - \mathbf{R}}{|\mathbf{r} - \mathbf{R}|^3} \right) \\ &= \frac{3(\mathbf{r} - \mathbf{R}) \boldsymbol{\mu} \cdot (\mathbf{r} - \mathbf{R}) - (\mathbf{r} - \mathbf{R})^2 \boldsymbol{\mu}}{|\mathbf{r} - \mathbf{R}|^5}. \end{aligned} \quad (9a)$$

The first term in Eq. (8) is required to satisfy the magnetic-charge-free Maxwell equation $\nabla \cdot \mathbf{B} = 0$:

$$\mathbf{B}_s = 4\pi \boldsymbol{\mu} \delta(\mathbf{r} - \mathbf{R}). \quad (9b)$$

If we ignore the latter for the moment, it is straightforward to work out the angular momentum from the field part,

$$\begin{aligned} \mathbf{J}_{f\epsilon\mu} &= \int (d\mathbf{r}) \mathbf{r} \times \frac{e}{4\pi c} \left(\frac{\mathbf{r}}{r^3} \times \nabla \mathbf{R} \right) \frac{\boldsymbol{\mu} \cdot (\mathbf{r} - \mathbf{R})}{|\mathbf{r} - \mathbf{R}|^3} \\ &= \frac{e}{4\pi c} \boldsymbol{\mu} \cdot \nabla \mathbf{R} \nabla \mathbf{R} \cdot \int (d\mathbf{r}) \left(\frac{\mathbf{r}\mathbf{r}}{r^3} - \frac{\mathbf{1}}{r} \right) \frac{1}{|\mathbf{r} - \mathbf{R}|}. \end{aligned} \quad (10)$$

The latter integral is evaluated to be $\pi R(\mathbf{1} + \mathbf{R}\mathbf{R}/R^2)$ and then the same result (7) follows.

However, for a point dipole, there is another contribution from the δ -function term in Eq. (8) which exactly cancels the above field part,

$$\mathbf{J}_{e\mu} = 0. \quad (11)$$

This term further gives a contribution to the field momentum:

$$\mathbf{P}_{e\mu} = -\frac{1}{c} \boldsymbol{\mu} \times \mathbf{E}(\mathbf{R}). \quad (12)$$

These results for the Gilbert dipole, Eqs. (7) and (5), and for the Ampère dipole, Eqs. (11) and (12), agree partly with those found by Furry [13], but he asserts without evidence that the Ampère angular momentum is unaltered from the Gilbert one, whereas his linear momentum seems to agree with ours.

BOOSTED FRAME

Our system, in the rest frame, in the Gilbert description, is defined by the electric and magnetic charge densities,

$$\rho_e = e\delta(\mathbf{r}), \quad \mathbf{j}_e = 0, \quad (13a)$$

$$\rho_m = -\boldsymbol{\mu} \cdot \nabla \delta(\mathbf{r} - \mathbf{R}), \quad \mathbf{j}_m = 0. \quad (13b)$$

Now consider a infinitesimally boosted frame, in which all particles are moving with velocity $\delta\mathbf{v} \ll c$. In such a frame the electric and magnetic fields are modified (Ref. [15], Sec. 10.3)

$$\delta\mathbf{E} = -\delta_{\text{coor}}\mathbf{E} - \frac{1}{c}\delta\mathbf{v} \times \mathbf{B}, \quad (14a)$$

$$\delta\mathbf{B} = -\delta_{\text{coor}}\mathbf{B} + \frac{1}{c}\delta\mathbf{v} \times \mathbf{E}, \quad (14b)$$

where

$$\delta_{\text{coor}} = \delta\mathbf{v}t \cdot \nabla + \frac{1}{c^2}\delta\mathbf{v} \cdot \mathbf{r} \frac{\partial}{\partial t}. \quad (15)$$

Then from Maxwell's equations with magnetic charge

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e, \quad -\nabla \times \mathbf{E} = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} + \frac{4\pi}{c} \mathbf{j}_m, \quad (16a)$$

$$\nabla \cdot \mathbf{B} = 4\pi\rho_m, \quad \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E} + \frac{4\pi}{c} \mathbf{j}_e, \quad (16b)$$

we deduce

$$\delta\rho_e = -\delta_{\text{coor}}\rho_e + \frac{1}{c^2}\delta\mathbf{v} \cdot \mathbf{j}_e, \quad (17a)$$

$$\delta\mathbf{j}_e = -\delta_{\text{coor}}\mathbf{j}_e + \delta\mathbf{v}\rho_e, \quad (17b)$$

$$\delta\rho_m = -\delta_{\text{coor}}\rho_m + \frac{1}{c^2}\delta\mathbf{v} \cdot \mathbf{j}_m, \quad (17c)$$

$$\delta\mathbf{j}_m = -\delta_{\text{coor}}\mathbf{j}_m + \delta\mathbf{v}\rho_m, \quad (17d)$$

To compute the torque in the boosted frame, we start from first principles, and use the microscopic Lorentz force density,

$$\mathbf{f} = \rho_e \mathbf{E} + \frac{1}{c} \mathbf{j}_e \times \mathbf{B} + \rho_m \mathbf{B} - \frac{1}{c} \mathbf{j}_m \times \mathbf{E}. \quad (18)$$

(Note the dual symmetry, replacing electric quantities by magnetic ones, and magnetic quantities by the negative of electric ones.) In the present problem, there is an induced magnetic charge density and current density,

$$\delta\rho_m = -\delta\mathbf{v}t \cdot \nabla\rho_m, \quad \delta\mathbf{j}_m = \delta\mathbf{v}\rho_m, \quad (19)$$

as well as an induced magnetic field acting on the dipole,

$$\delta\mathbf{B} = \frac{1}{c}\delta\mathbf{v} \times \mathbf{E}. \quad (20)$$

Thus the torque on the dipole in the moving frame is

$$\begin{aligned} \boldsymbol{\tau} &= \int (d\mathbf{r}) \mathbf{r} \times \left(\rho_m \delta\mathbf{B} - \frac{1}{c} \delta\mathbf{j}_m \times \mathbf{E} \right) \\ &= \int (d\mathbf{r}) \mathbf{r} \times \left(\rho_m \frac{\delta\mathbf{v}}{c} \times \mathbf{E} - \frac{\delta\mathbf{v}}{c} \rho_m \times \mathbf{E} \right) = 0, \end{aligned} \quad (21)$$

that is, the two terms exactly cancel. In precisely the same way, we can show that the torque on the electric charge vanishes as well. Thus there is no torque and no violation of relativistic invariance. The reason for the apparent anomaly was the invalid use of Eq. (1), which applies only for constant fields, and cannot be applied indiscriminately. In fact, a discussion equivalent to this is given in Problem 3, Chapter 4, in Ref. [15].

So there is no anomaly for a Gilbert dipole. What happens for an Ampèrian one? In that case we have no magnetic charge or current, so the torque is

$$\begin{aligned} \boldsymbol{\tau} &= \int (d\mathbf{r}) \mathbf{r} \times \left[\frac{\delta\mathbf{v}}{c^2} \cdot \mathbf{j}_e \mathbf{E} + \frac{1}{c} \mathbf{j}_e \times \left(\frac{\delta\mathbf{v}}{c} \times \mathbf{E} \right) \right] \\ &= -\frac{\delta\mathbf{v}}{c^2} \times \int (d\mathbf{r}) \mathbf{r} \mathbf{j} \cdot \mathbf{E}. \end{aligned} \quad (22)$$

Now, ignoring an electric quadrupole term [Chapter 32, Ref. [15]], we may replace $\mathbf{r} \mathbf{j} \rightarrow \frac{1}{2}(\mathbf{r} \mathbf{j} - \mathbf{j} \mathbf{r})$, and so we evaluate

$$\boldsymbol{\tau} = \frac{\delta\mathbf{v}}{c} \times (\boldsymbol{\mu} \times \mathbf{E}), \quad (23)$$

which uses the definition of the magnetic moment,

$$\boldsymbol{\mu} = \frac{1}{2c} \int (d\mathbf{r}) \mathbf{r} \times \mathbf{j}(\mathbf{r}). \quad (24)$$

The torque on the dipole does not vanish.

But now there is a contribution from the field angular momentum in the boosted frame:

$$\delta\mathbf{J} = \frac{1}{4\pi c} \int (d\mathbf{r}) \mathbf{r} \times (\delta\mathbf{E} \times \mathbf{B} + \mathbf{E} \times \delta\mathbf{B}). \quad (25)$$

This is simply seen to be

$$\delta\mathbf{J} = \delta\mathbf{v}t \times \mathbf{P}_{e\mu} + \text{constant}, \quad (26)$$

where the time-dependent term arises from the coordinate variation. This angular momentum is expressed in

terms of the momentum of the field in the rest frame. The latter is zero for a Gilbert dipole, but is given by Eq. (12) for a point dipole. Thus, in the latter case,

$$\boldsymbol{\tau} + \frac{d}{dt} \delta\mathbf{J} = 0, \quad (27)$$

So again there is no inconsistency.

CONCLUSIONS

In this note we have examined the question of the torque on a magnetic dipole due to an electric charge. There is, in contrast to claims in the literature [1], no contradiction with special relativity, which could hardly be otherwise, since the equations of electromagnetism are the origin of Einsteinian relativity. The details of how this consistency is achieved depends on the model of the dipole. If we apply the simple fiction that the dipole consists of a monopole-antimonopole pair, the torque, computed from the microscopic Lorentz force density, vanishes in the boosted frame. If the model of the dipole is consistent with the Ampèrian hypothesis that all magnetic effects emerge from moving electric charges or changing electric fields, the torque is not zero, but is balanced by the change in the field momentum. So there is no mechanical torque on the dipole.

Although there has been a chorus of critiques of the claim of Ref. [1], none has seemed definitive to us. The arguments seem to rely on special models and approximate expressions. We have not adopted any specific model, but considered either the fictional but useful picture of a Gilbert dipole composed of a magnetic monopole-antimonopole pair, or that of a generic infinitesimal Ampèrian current loop. On that basis the consistency of the theory, in particular, the validity of the microscopic Lorentz force law, is established simply from Maxwell's equations without further assumptions.

This work was supported by grants from the US National Science Foundation and the US Department of Energy. We thank Elom Abalo, Nathan Beck, Nathan Edmonson, Prachi Parashar, K. V. Shajesh, and Jimmy Wu for helpful discussions. GM thanks the NSF-funded REU program at OU for summer support.

* Electronic address: milton@nhn.ou.edu

† Electronic address: giulioemeille@gmail.com

[1] M. Mansuripur, Phys. Rev. Lett. **108**, 193901 (2012).

[2] M. Mansuripur, Optics Comm. **284**, 594 (2011).

[3] A. Einstein and J. Laub, Ann. Phys. (Leipzig) **331**, 532 (1908).

[4] A. Einstein and J. Laub, Ann. Phys. (Leipzig) **331**, 541 (1908).

- [5] Adrian Cho, “Textbook Electrodynamics May Contradict Relativity,” *Science* **336**, 404 (27 April 2012) [DOI: 10.1126/science.336.6080.404].
- [6] C. S. Unnikrishnan, arXiv:1205.1080v2.
- [7] D. A. T. Vanzella, arXiv:1205.1502v2.
- [8] D. J. Griffiths and V. Hnizdo, arXiv:1205.4646.
- [9] D. J. Cross, arXiv:1205.5451v2.
- [10] K. T. McDonald, “Mansuripur’s paradox,” www.physics.princeton.edu/~mcdonald/examples/mansuripur.pdf. This reference contains extensive citations of the literature.
- [11] T. H. Boyer, arXiv:1206.5322.
- [12] M. Brachet and E. Tirapegul, arXiv:1207.4613.
- [13] W. H. Furry, *Am. J. Phys.* **27**, 621 (1969).
- [14] D. Babson, S. P. Reynolds, R. Bjorkquist, and D. J. Griffiths, *Am. J. Phys.* **77**, 826 (2009).
- [15] J. Schwinger, L. L. DeRaad, Jr., K. A. Milton, and W.-y. Tsai, *Classical Electrodynamics* (Perseus, New York, 1998).
- [16] H. Poincaré, *Compt. Rendus* **123**, 330 (1896).
- [17] J. J. Thomson, *Philos. Mag.* **8**, 331 (1904).